INTEGRATION

OPERATORS:-

1. POWER $\quad (\square)^n \longrightarrow \dfrac{(\square)^{n+1}}{n+1}$

2. TRIG
   - $\sin\square \longrightarrow -\cos\square$
   - $\cos\square \longrightarrow \sin\square$
   - $\sec^2\square \longrightarrow \tan\square$

3. EXP $\quad e^{\square} \longrightarrow e^{\square}$

4. LOG $\quad \dfrac{\square'}{\square} \longrightarrow \ln\square$

5. INVERSE TAN: $\quad \dfrac{\square'}{1+\square^2} \longrightarrow \tan^{-1}\square$

RULES FOR INTEGRATION

1) YOU ARE NOT ALLOWED TO INTEGRATE AN OPERATOR UNLESS DIFFERENTIATION OF BOX $\square'$ IS PRESENT OUTSIDE THE OPERATOR (Nothing else should be present outside)

2) ONCE THIS CONDITION IS FULFILLED THREE THINGS DISAPPEAR

$$\int \square' \qquad dx$$

AND YOU ARE ALLOWED TO INTEGRATE OPERATOR.

1) $\int (x+3)^5 \, dx$

$$\int 1 \, (\boxed{x+3})^5 \, dx \qquad \square = x+3, \quad \square' = 1$$

$$\frac{(x+3)^6}{6}$$

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$$\int \square' \qquad dx$$

AND YOU ARE ALLOWED TO INTEGRATE OPERATOR.

2) $\int (2x+5)^3 \, dx$

RULES FOR INTEGRATION

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2) ONCE THIS CONDITION IS FULFILLED THREE THINGS DISAPPEAR

$\int \square' \quad dx$

AND YOU ARE ALLOWED TO INTEGRATE OPERATOR.

$$\frac{1}{2}\int 2(\boxed{2x+5})^{3}\,dx \qquad \square = 2x+5 \qquad \square' = 2$$

$$\frac{1}{2}\,\frac{(2x+5)^{4}}{4}$$

$$\frac{(2x+5)^{4}}{8}$$

3) $\int x(4x^2+7)^5\,dx$

RULES FOR INTEGRATION

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2) ONCE THIS CONDITION IS FULFILLED THREE THINGS DISAPPEAR

$\int \square' \quad dx$

AND YOU ARE ALLOWED TO INTEGRATE OPERATOR.

$$\frac{1}{8}\int 8x(\boxed{4x^2+7})^{5}\,dx \qquad \square = 4x^2+7 \qquad \square' = 8x$$

$$\frac{1}{8}\,\frac{(4x^2+7)^{6}}{6}$$

$$\frac{(4x^2+7)^{6}}{48}$$

YOU ARE ALLOWED TO INTRODUCE / REMOVE CONSTANTS WHILE COMPLETING DIFFERENTIATION OF BOX OUTSIDE THE OPERATOR.

YOU ARE NOT ALLOWED TO TO INTRODUCE / REMOVE A VARIABLE TERM TO COMPLETE DIFFERENTIATION OF BOX OUTSIDE THE OPERATOR.

4) $\int \sin 3x \, dx$

$\frac{1}{3}\int 3 \sin \boxed{3x} \, dx$ $\quad \square = 3x$, $\square' = 3$

$\frac{1}{3}\left(-\cos(3x)\right)$

$\frac{-\cos(3x)}{3}$

5) $\int \cos\left(2x+\frac{\pi}{3}\right) dx$

$\frac{1}{2}\int 2 \cos\left(\boxed{2x+\frac{\pi}{3}}\right) dx$ $\quad \square = 2x+\frac{\pi}{3}$, $\square' = 2$

$\frac{1}{2}\sin\left(2x+\frac{\pi}{3}\right)$

$\boxed{6}$ $\int \sec^2 4x \, dx$

$\frac{1}{4}\int 4 \sec^2 \boxed{4x} \, dx$ $\quad \square = 4x$, $\square' = 4$

$\frac{1}{4}\tan 4x$

7) $\int e^{2x+5} dx$

$\frac{1}{2}\int 2 e^{\boxed{2x+5}} dx$ $\quad \square = 2x+5$, $\square' = 2$

$\frac{1}{2}e^{2x+5}$

MULTIPLE OPERATOR

$\int x^2(x^3+1)^7 dx$

$\frac{1}{3}\int 3x^2\left(\boxed{x^3+1}\right)^7 dx$ $\quad \square = x^3+1$, $\square' = 3x^2$

$\frac{1}{3}\frac{(x^3+1)^8}{8}$

$\int \cos x \sin^3 x \, dx$

$\int \cos x \left(\boxed{\sin x}\right)^3 dx$ $\quad \square = \sin x$, $\square' = \cos x$

$\frac{(\sin x)^4}{4}$

$$\int \sin^3 x \cos x \, dx$$

$$\int (\boxed{\sin x})^3 \cos x \, dx$$

$\square = \sin x$

$\square' = \cos x$

$$\frac{(\sin x)^4}{4}$$

$$\int \sin^3 x \, \underline{\cos \boxed{x}} \, dx$$

$\square = x$

Reject $\square' = 1$

$$\int \sec^2 x \tan^3 x \, dx$$

$$\int \sec^2 x (\boxed{\tan x})^3 \, dx$$

$\square = \tan x$

$\square' = \sec^2 x$

$$\frac{(\tan x)^4}{4}$$

$$\int \sec^2 x \tan x \, dx$$

$$\int \sec^2 x (\boxed{\tan x})^1 \, dx$$

$\square = \tan x$

$\square' = \sec^2 x$

$$\frac{(\tan x)^2}{2}$$

NOTE: IF TAN & $\sec^2$ are together, Power of tan becomes operator.

$$\boxed{\text{TAN \& SEC}^2 \text{ ARE COUSINS}}$$

1. TRIG: $1 + \tan^2 x = \sec^2 x$
2. DIFF: $\tan \square \xrightarrow{\text{DIFF}} \sec^2 \square \times \square'$
3. INTEG: $\sec^2 \square \xrightarrow{\text{INTEG}} \tan \square$
4. INTEG: WHEN TAN & SEC^2 are together, power of tan is operator.
5. Integ: odd / even powers of tan.

   $1 + \tan^2 x = \sec^2 x$

$\boxed{3}$ $\int \cos x \, e^{\sin x} \, dx$

$\int \cos x \, e^{\boxed{\sin x}} \, dx$ $\quad \square = \sin x$

$\square' = \cos x$

$e^{\sin x}$

HOW TO CHOOSE AN OPERATOR :

IF WHILE COMPLETING "DIFFERENTIATION OF BOX" $\square'$ OUTSIDE THE OPERATOR , YOU NEED TO INTRODUCE / REMOVE A VARIABLE TERM, THAT OPERATOR IS REJECTED.

$\int \sin x \, e^{\cos x} \, dx$

$\int \sin \boxed{x} \, e^{\cos x} \, dx$

$\square = x$

$\square' = 1$

To complete $\square'$ outside we need to remove $e^{\cos x}$

OPERATOR REJECT

$-1 \int -\sin x \, e^{\boxed{\cos x}} \, dx$

$\square = \cos x$

$\square' = -\sin x$

$-1 \; e^{\cos x}$

$$\int \sin 2x \cos^3 2x \, dx$$

$$\int \sin \boxed{2x} \cos^3 2x \, dx$$

$\square = 2x$

$\square' = 2$

To complete $\square'$, you need to remove $\cos^3 2x$

REJECT.

$$\frac{1}{-2} \int -2\sin 2x \left(\boxed{\cos 2x}\right)^3 dx$$

$\square = \cos 2x$

$\square' = -\sin 2x \times 2$

$$\frac{1}{-2} \frac{(\cos 2x)^4}{4}$$

3) $$\int x^3 e^{4x^4+2} \, dx$$

$$\int \boxed{x}^3 e^{4x^4+2} \, dx$$

$\square = x$

$\square' = 1$

we need to remove e^{4x^4+2} (REJECT)

$$\frac{1}{16} \int 16x^3 e^{\boxed{4x^4+2}} \, dx$$

$\square = 4x^4+2$

$\square' = 16x^3$

$$\frac{1}{16} e^{4x^4+2}$$

$\int x \sin x^2 \, dx$	$\int x \sin x \, dx$
$\int \boxed{x}' \sin x^2 \, dx$ $\square = x$, $\square' = 1$ REJECT.	$\int \boxed{x}' \sin x \, dx$ $\square = x$, $\square' = 1$ REJECT.
$\frac{1}{2}\int 2x \sin \boxed{x^2} \, dx$ $\square = x^2$, $\square' = 2x$ $\frac{1}{2}\left(-\cos x^2\right)$	$\int x \sin \boxed{x} \, dx$ $\square = x$, $\square' = 1$ Reject. PRODUCT RULE (BY-PARTS)

OPERATORS

- SINGLE OPERATOR
  - Operator accepted → SINGLE STEP INTEGRATION
  - Operator rejected → EVEN/ODD powers of sin/cos/tan
- DOUBLE OPERATOR
  - ONE Operator accepted → SINGLE STEP INTEGRATION
  - BOTH OPERATOR REJECT → Integration By-Parts / PRODUCT RULE (can go upto 2 pages)

→ INTEGRATION BY SUBSTITUTION IS GIVEN IN THE QUESTION.

SINGLE OPERATOR REJECT

$\downarrow$

Even/odd powers of sin/cos

EVEN

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

ODD

STEP1: Split power.

1 · (rest)

$\downarrow$

STEP2 Apply $\sin^2\theta + \cos^2\theta = 1$

EVEN/ODD POWERS OF TAN

EVEN

$1 + \tan^2 x = \sec^2 x$

ODD

Split 1 · rest

$\downarrow$

$1 + \tan^2 x = \sec^2 x$

1) $\int \sin^2 x \, dx$ $\qquad$ $\int (\boxed{\sin x})^2 dx$

$\square = \sin x$

$\square' = \cos x$

reject.

$\int \frac{1-\cos 2x}{2} dx$ $\qquad$ $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

$\frac{1}{2}\int (1-\cos 2x)\,dx$

$\frac{1}{2}\left[\int 1\,dx - \frac{1}{2}\int 2\cos \boxed{2x}\,dx\right]$ $\qquad$ $\square = 2x$, $\square' = 2$

$\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]$

2) $\int \cos^2 x \, dx$ $\qquad$ $\int (\boxed{\cos x})^2 dx$

$\square = \cos x$

$\square' = -\sin x$

reject.

$\int \frac{1+\cos 2x}{2} dx$

$\frac{1}{2}\left[\int 1\,dx + \frac{1}{2}\int 2\cos \boxed{2x}\,dx\right]$ $\qquad$ $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\square = 2x$

$\square' = 2$

$\frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]$

3) $\int \sin^3 x \, dx$

split power.

$\int \sin x \cdot \sin^2 x \, dx$ $\quad\quad \sin^2 x + \cos^2 x = 1$

$\int \sin x (1 - \cos^2 x) \, dx$

$\int 1 \sin \boxed{x} \, dx - \int \sin x \cos^2 x \, dx$

$-\cos x - (-1) \int -\sin x \left(\boxed{\cos x}\right)^2 dx$ $\quad\quad \square = \cos x$, $\square' = -\sin x$

$$\boxed{-\cos x + \frac{(\cos x)^3}{3}}$$

4) $\int \cos^3 x \, dx$

split power

$\int \cos x \cdot \cos^2 x \, dx$ $\quad\quad \sin^2 x + \cos^2 x = 1$

$\int \cos x (1 - \sin^2 x) \, dx$

$\int 1 \cos \boxed{x} \, dx - \int \cos x \sin^2 x \, dx$

$\sin x - \int \cos x \left(\boxed{\sin x}\right)^2 dx$ $\quad\quad \square = \sin x$, $\square' = \cos x$

$$\sin x - \frac{(\sin x)^3}{3}$$

New operator

$$\int \frac{\square'}{\square}\,dx = \ln \square$$

1) $\int \tan x\,dx$

$$-1\int \frac{-\sin x}{\boxed{\cos x}}\,dx$$

$\square = \cos x$

$\square' = -\sin x$

$-1 \ln|\cos x|$

2) $\int \cot x\,dx$

$$\int \frac{\cos x}{\boxed{\sin x}}\,dx$$

$\square = \sin x$

$\square' = \cos x$

$\ln|\sin x|$

3) $\int \frac{1}{2x+3}\,dx$

$$\frac{1}{2}\int \frac{2\times 1}{\boxed{2x+3}}\,dx$$

$\square = 2x+3$

$\square' = 2$

$\frac{1}{2}\ln(2x+3)$

4) $\int \frac{1}{\boxed{(2x+3)^2}}\,dx$

$\square = (2x+3)^2$

$\square' = 2(2x+3)'(2)$

$\square' = 4(2x+3)$

we cannot introduce this.

$$\frac{1}{2}\int 2(\boxed{2x+3})^{-2}\,dx$$

$\square = 2x+3$

$\square' = 2$

$$\frac{1}{2}\,\frac{(2x+3)^{-1}}{-1}$$

$$-\frac{1}{2(2x+3)}$$

Let's Try power operator here.

$$\int \frac{1}{2x+3}\,dx$$

$$\frac{1}{2}\int 2\,(\boxed{2x+3})^{-1}\,dx \qquad \square = 2x+3$$

$$\square' = 2$$

$$\frac{1}{2}\,\frac{(2x+3)^{-1+1}}{0}$$

→ Alert: This is where you should have used $\ln\square$ operator.

ODD/EVEN POWERS OF TAN

1) $$\int \tan^2 x\,dx \qquad 1+\tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\int (\sec^2 x - 1)\,dx$$

$$\int 1\,\sec^2\boxed{x}\,dx - \int 1\,dx$$

$$\tan x - x$$

2) $$\int \tan^3 x\,dx$$

$$\int \tan x \cdot \tan^2 x\,dx \qquad 1+\tan^2 x = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\int \tan x\,(\sec^2 x - 1)\,dx$$

$$\int (\tan x \sec^2 x - \tan x)\,dx$$

$$\int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$\int (\boxed{\tan x}) \sec^2 x \, dx - (-1)\int \frac{-\sin x}{\boxed{\cos x}} \, dx$$

$\square = \tan x$

$\square' = \sec^2 x$

$\square = \cos x$

$\square' = -\sin x$

$$\boxed{\frac{(\tan x)^2}{2} + \ln|\cos x|}$$

WHEN BOTH OPERATORS ARE REJECTED
PRODUCT RULE (BY-PARTS)

$$\int uv \, dx = u \int v \, dx - \int \left[\frac{du}{dx} \times \int v \, dx \right] dx$$

$$(u)(\text{integ of } v) - \int \left[\begin{matrix}\text{Diff}\\ \text{of } u\end{matrix} \times \begin{matrix}\text{Integ}\\ \text{of } v\end{matrix} \right] dx$$

NOTE: In Integration This product Rule (By parts) is not universal. This ONLY works when both operators are rejected.

$\boxed{1}$

$$\int \underbrace{x}_{u}\ \underbrace{\sin x}_{v}\ dx \qquad\qquad \int \boxed{x}^{1} \sin \boxed{x} \ dx$$

$\square = x$, $\square' = 1$ Reject; $\square = x$, $\square' = 1$ Reject

Diff of u	Integ of v
$x \longrightarrow \boxed{1}$	$\int {}^{1}\sin \boxed{x}\, dx$
	$\boxed{-\cos x}$

$$u\int v\,dx - \int\left[\frac{du}{dx} \times \int v\,dx\right] dx$$

$$(x)(-\cos x) - \int\left[1 \times (-\cos x)\right] dx$$

$$-x\cos x + \int 1 \cos \boxed{x}\ dx$$

$$\boxed{-x\cos x + \sin x}$$

2)

$$\int \underbrace{x}_{u}\, \underbrace{e^{3x}}_{v}\, dx \qquad\qquad \int \boxed{x}^{1} e^{\boxed{3x}}\, dx$$

$\square = x$, $\square' = 1$ reject; $\square = 3x$, $\square' = 3$ reject

Diff of u	Integ of v
$x \longrightarrow 1$	$\frac{1}{3}\int 3\, e^{\boxed{3x}}\, dx = \frac{1}{3} e^{3x}$

$$u\int v\,dx - \int\left[\frac{du}{dx} \times \int v\,dx\right] dx$$

$$(x)\left(\frac{e^{3x}}{3}\right) - \int\left[1 \times \frac{e^{3x}}{3}\right] dx$$

$$\frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}\, dx$$

$$\frac{xe^{3x}}{3} - \frac{1}{3}\left(\frac{e^{3x}}{3}\right)$$

$$\boxed{\frac{xe^{3x}}{3} - \frac{e^{3x}}{9}}$$

HOW TO DECIDE u and v for BY-PARTS:

$\boxed{\text{STRICT RULE}}$ $\ln\square$ are always u

$\boxed{\text{PREFERRABLE RULE}}$ $e^{\square}$, $\sin\square$, $\cos\square$ are preferred to be taken as v

$\boxed{\text{GENERAL RULE}}$ If any terms reduces to zero after doing repeated diff, it is ideal for u

$x \longrightarrow 1 \longrightarrow 0$

$x^3 \longrightarrow 3x^2 \longrightarrow 6x \longrightarrow 6 \longrightarrow 0$

3) $$\int \underbrace{x^3}_{v}\,\underbrace{\ln x}_{u}\,dx$$

Diff of u	Integ of v
$\ln x \longrightarrow \frac{1}{x}$	$\int x^3\,dx = \frac{x^4}{4}$

$$u\int v\,dx - \int\left[\frac{du}{dx} \times \int v\,dx\right]dx$$

$$(\ln x)\left(\frac{x^4}{4}\right) - \int\left[\frac{1}{x} \times \frac{x^4}{4}\right]dx$$

$$(4) \int [x \quad 4]$$

$$\frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 \, dx$$

$$\boxed{\frac{x^4 \ln x}{4} - \frac{1}{4}\left(\frac{x^4}{4}\right)}$$

Integrate

With limits	Without limits
No need for +c	+c is mandatory.

SPECIAL CASE: MEMORIZE.

$$\int \ln x \, dx$$

Use BY-PART (PRODUCT)

$$\int \underbrace{1}_{v} \cdot \underbrace{\ln x}_{u} \, dx$$

Diff of u	integ of v
$\ln x \rightarrow \frac{1}{x}$	$\int 1 \, dx = x$

$$u\int v\,dx - \int\left[\frac{du}{dx} \times \int v\,dx\right]dx$$

$$(\ln x)(x) - \int\left[\frac{1}{x} \times x\right]dx$$

$$x \ln x - \int 1 \, dx$$

$$\boxed{x \ln x - x}$$